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VANISHING OF THE THIRD SIMPLICIAL COHOMOLOGY GROUP OF $l^1(\mathbf{Z}_+)$

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ABSTRACT. We show that $\mathcal{H}^3(l^1(\mathbf{Z}_+), l^1(\mathbf{Z}_+)') = 0$. We first use the Connes-Tzygan exact sequence to prove that this is equivalent to the vanishing of the third cyclic cohomology group $\mathcal{HC}^3(\mathcal{I}, \mathcal{I}')$, where $\mathcal{I}$ is the non-unital Banach algebra $l^1(\mathbf{N})$, and then prove that $\mathcal{HC}^3(\mathcal{I}, \mathcal{I}') = 0$.

1. Preliminaries

It has been known for some time [1] that $l^1(\mathbf{Z}_+)$, the unital semigroup algebra of $\mathbf{N}$, is not weakly amenable, that is $\mathcal{H}^1(l^1(\mathbf{Z}_+), l^1(\mathbf{Z}_+)') \neq 0$. This may lead one to believe that $\mathcal{H}^n(l^1(\mathbf{Z}_+), l^1(\mathbf{Z}_+)')$, the higher simplicial cohomology groups, are also non-zero for $n \geq 2$. However, Johnson showed in [7] that the alternating cohomology of this algebra vanishes in all dimensions strictly greater than 1. Then, in a systematic calculation of second cohomology groups, Dales and Duncan [3, Theorem 3.2] showed that the second simplicial cohomology of $l^1(\mathbf{Z}_+)$ is trivial. This leads to the conjecture that all the simplicial cohomology groups of $l^1(\mathbf{Z}_+)$ vanish for $n \geq 2$.

In this paper, we show that the third simplicial cohomology group of $l^1(\mathbf{Z}_+)$ vanishes. The proof is harder than one might expect and proceeds by way of a reduction of the $\mathcal{H}^3(l^1(\mathbf{Z}_+), l^1(\mathbf{Z}_+)')$ question to a question about some cyclic *approximate* 2-cocycles (which are cyclic 2-cochains having a small coboundary).

In the algebraic analogue of this theorem, where the algebra in question is the polynomial ring $\mathbf{C}[X]$, the polynomial ring has dimension 1 as a bimodule over itself and so its second and higher cohomology groups vanish for any coefficient bimodule. The same cannot hold for the algebra $l^1(\mathbf{Z}_+)$. In fact, Dales and Duncan [3] show that $\mathcal{H}^2(l^1(\mathbf{Z}_+), c_0(\mathbf{Z}_+)') \neq 0$, and so not all second cohomology groups of $l^1(\mathbf{Z}_+)$ are trivial, even with coefficients in dual modules.

We now recall some basic results and introduce our notation. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}'$ be a Banach $\mathcal{A}$-bimodule in the usual way. An n-cochain is a bounded n-linear map T from $\mathcal{A}$ to $\mathcal{A}'$, which we denote by $T \in C^n(\mathcal{A}, \mathcal{A}')$. The

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map $\delta^n : C^n(\mathcal{A},\mathcal{A}') \to C^{n+1}(\mathcal{A},\mathcal{A}')$ is defined by

$$\begin{aligned}(\delta^n T)(a_1,\dots,a_{n+1})(a_0) &= T(a_2,a_3,\dots,a_{n+1})(a_0a_1)\\ &\quad - T(a_1a_2,a_3,\dots,a_{n+1})(a_0)\\ &\quad + T(a_1,a_2a_3,a_4,\dots,a_{n+1})(a_0)+\dots\\ &\quad + (-1)^n T(a_1,\dots,a_{n-1},a_na_{n+1})(a_0)\\ &\quad + (-1)^{n+1}T(a_1,\dots,a_n)(a_{n+1}a_0)\,.\end{aligned}$$

The n-cochain T is an *n-cocycle* if $\delta^nT=0$ and it is an *n-coboundary* if $T=\delta^{n-1}S$ for some $S\in C^{n-1}(\mathcal{A},\mathcal{A}')$. Throughout this paper we use *normalized cochains*, which are cochains T such that $T(a_1,a_2,\dots,a_n)(a_0)=0$ whenever one of the variables a_1, a_2, …, a_n is a multiple of the identity. The linear space of all n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A},\mathcal{A}')$, and the linear space of all n-coboundaries is denoted by $\mathcal{B}^n(\mathcal{A},\mathcal{A}')$. We also recall that $\mathcal{B}^n(\mathcal{A},\mathcal{A}')$ is included in $\mathcal{Z}^n(\mathcal{A},\mathcal{A}')$ and that the n^{th} cohomology group $\mathcal{H}^n(\mathcal{A},\mathcal{A}')$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A},\mathcal{A}')=\frac{\mathcal{Z}^n(\mathcal{A},\mathcal{A}')}{\mathcal{B}^n(\mathcal{A},\mathcal{A}')}.$$

It is a standard fact [4, page 75] that normalized cochains define the same cohomology as do cochains.

The n-cochain T is called *cyclic* if

$$T(a_1,a_2,\dots,a_n)(a_0)=(-1)^nT(a_0,a_1,\dots,a_{n-1})(a_n),$$

and we denote the linear space of all cyclic n-cochains by $CC^n(\mathcal{A},\mathcal{A}')$. It is well known (see [5]) that the cyclic cochains $CC^n(\mathcal{A},\mathcal{A}')$ form a subcomplex of $C^n(\mathcal{A},\mathcal{A}')$, that is $\delta^n : CC^n(\mathcal{A},\mathcal{A}')\to CC^{n+1}(\mathcal{A},\mathcal{A}')$, and so we have cyclic versions of the spaces defined above, which we denote by $\mathcal{BC}^n(\mathcal{A},\mathcal{A}')$, $\mathcal{ZC}^n(\mathcal{A},\mathcal{A}')$ and $\mathcal{HC}^n(\mathcal{A},\mathcal{A}')$. Note that it is usual to denote the cyclic cohomology group by $\mathcal{HC}^n(\mathcal{A})$, as there is only one bimodule used, namely $\mathcal{A}'$. For the same reason, we will often denote $CC^n(\mathcal{A},\mathcal{A}')$ by $CC^n(\mathcal{A})$, $\mathcal{BC}^n(\mathcal{A},\mathcal{A}')$ by $\mathcal{BC}^n(\mathcal{A})$ and $\mathcal{ZC}^n(\mathcal{A},\mathcal{A}')$ by $\mathcal{ZC}^n(\mathcal{A})$.

The n-cochain T is *even* if $T=T^{\text{op}}$ and *odd* if $T=-T^{\text{op}}$, where

$$T^{\text{op}}(a_1,a_2,\dots,a_n)(a_0)=(-1)^{k_n}T(a_n,a_{n-1},\dots,a_1)(a_0)$$

and $k_n=\frac{n(n+1)}{2}+1$. (The value of k_n is determined by the requirement that $\delta^n(T^{\text{op}})=(\delta^nT)^{\text{op}}$.) With these definitions, it is easy to check that $T_+=\frac{1}{2}(T+T^{\text{op}})$ is even, $T_-=\frac{1}{2}(T-T^{\text{op}})$ is odd, and we note that $T=T_++T_-$.

Lemma 1.1. *Let T be an n-cochain, and let T^{op} be defined as above. Then the following hold.*

1. *T is cyclic iff T^{op} is cyclic.*
2. *T is cyclic iff T_+ and T_- are cyclic.*
3. *If $\|\delta^nT\|\le M$, then $\|\delta^nT_-\|\le M$ and $\|\delta^nT_+\|\le M$.*

Proof. We have, using the definition of T^{op} and the fact that T is cyclic, that

$$\begin{aligned}T^{\text{op}}(a_1,a_2,\dots,a_n)(a_0) &= (-1)^{k_n}T(a_n,a_{n-1},\dots,a_1)(a_0)\\ &= (-1)^{k_n}(-1)^nT(a_{n-1},a_{n-2},\dots,a_0)(a_n)\\ &= (-1)^{2k_n}(-1)^nT^{\text{op}}(a_0,a_1,\dots,a_{n-1})(a_n)\\ &= (-1)^nT^{\text{op}}(a_0,a_1,\dots,a_{n-1})(a_n),\end{aligned}$$

and so T^{op} is cyclic. The converse obviously holds, as $(T^{\mathrm{op}})^{\mathrm{op}} = T$.

The second statement easily follows from the first, and the last immediately follows from the definition and the fact that $\|\delta^n T\| = \|\delta^n T^{\mathrm{op}}\|$. □

2. Reduction to cyclic cohomology

Throughout this section, $\mathcal{A} = l^1(\mathbf{Z}_+)$, where

$$l^1(\mathbf{Z}_+) = \left\{ f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

with norm $\|f\| = \sum_{n=0}^{\infty} |a_n|$ and multiplication given by the usual convolution multiplication on $\mathbf{Z}_+$. We let $\mathcal{I}$ be the ideal of $\mathcal{A}$ given by

$$\mathcal{I} = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A} : a_0 = 0 \right\},$$

and we let $\mathbf{C}_0$ be the 1-dimensional bimodule given by $\mathbf{C}_0 = \mathcal{A}/\mathcal{I}$.

Proposition 2.1. *With the notation as above and for $n \geq 2$, $\mathcal{H}^n(\mathcal{A}, \mathcal{A}')$ is isomorphic to $\mathcal{H}^n(\mathcal{A}, \mathcal{I}')$, and hence isomorphic to $\mathcal{H}^n(\mathcal{I}, \mathcal{I}')$.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathbf{C}_0 \to 0\,.$$

The dual of this short exact sequence, which is also a short exact sequence, is

$$0 \leftarrow \mathcal{I}' \leftarrow \mathcal{A}' \leftarrow \mathbf{C}_0' \leftarrow 0\,.$$

This gives us (see [4, Section III, Theorem 3.2]) the long exact sequence of cohomology

$$\ldots \to \mathcal{H}^n(\mathcal{A}, \mathbf{C}_0) \to \mathcal{H}^n(\mathcal{A}, \mathcal{A}') \to \mathcal{H}^n(\mathcal{A}, \mathcal{I}') \to \mathcal{H}^{n+1}(\mathcal{A}, \mathbf{C}_0) \to \ldots\,.$$

By [5, Remark 12] we have $\mathcal{H}^{n+1}(\mathcal{I}, \mathbf{C}_0) = \mathrm{Ext}^n_{\mathcal{I}}(\mathbf{C}_0, \mathcal{I}')$ for all $n > 0$. As in [5, Example 21], $\mathcal{I}$ is a projective, and hence flat $\mathcal{I}$-module, and thus $\mathrm{Ext}^n_{\mathcal{I}}(\mathbf{C}_0, \mathcal{I}') = 0$ for all $n > 0$. Therefore $\mathcal{H}^n(\mathcal{I}, \mathbf{C}_0) = 0$ for all $n > 1$.

As we can use normalized cochains to calculate cohomology, we have $\mathcal{H}^n(\mathcal{I}, \mathbf{C}_0) = \mathcal{H}^n(\mathcal{A}, \mathbf{C}_0)$. Thus $\mathcal{H}^n(\mathcal{A}, \mathbf{C}_0) = 0$ for all $n \geq 2$, and we can deduce from the long exact sequence of cohomology given above that $\mathcal{H}^n(\mathcal{A}, \mathcal{A}') = \mathcal{H}^n(\mathcal{A}, \mathcal{I}')$ for all $n \geq 2$. Using normalized cochains again gives $\mathcal{H}^n(\mathcal{A}, \mathcal{I}') = \mathcal{H}^n(\mathcal{I}, \mathcal{I}')$, which completes the proof. □

The next lemma is essentially [5, Example 21], with the disc algebra replaced by $l^1(\mathbf{Z}_+)$.

Lemma 2.2. *The portion of the Connes-Tzygan exact sequence beginning with $\mathcal{HC}(\mathcal{I}) \to \mathcal{H}^2(\mathcal{I}, \mathcal{I}')$ exists; that is,*

$$\mathcal{HC}^2(\mathcal{I}) \to \mathcal{H}^2(\mathcal{I}, \mathcal{I}') \to \mathcal{HC}^1(\mathcal{I}) \to \mathcal{HC}^3(\mathcal{I}) \to \mathcal{H}^3(\mathcal{I}, \mathcal{I}') \to \mathcal{HC}^2(\mathcal{I}) \to \ldots\,.$$

Proof. As observed in the proof of Proposition 2.1, $\mathrm{Ext}^n_{\mathcal{I}}(\mathbf{C}_0, \mathcal{I}') = 0$ for all $n > 0$. The conclusion then follows from the proof of [5, Theorem 11]. □

Theorem 2.3. $\mathcal{H}^3(\mathcal{A}, \mathcal{A}') = \mathcal{HC}^3(\mathcal{I})$.

Proof. Proposition 2.1 implies that $\mathcal{H}^3(\mathcal{A},\mathcal{A}') = \mathcal{H}^3(\mathcal{I},\mathcal{I}')$, and thus we only need to show that $\mathcal{H}^3(\mathcal{I},\mathcal{I}') = \mathcal{HC}^3(\mathcal{I})$. Lemma 2.2 implies the existence of the following part of the Connes-Tzygan exact sequence:

$$\mathcal{HC}^1(\mathcal{I}) \to \mathcal{HC}^3(\mathcal{I}) \to \mathcal{H}^3(\mathcal{I},\mathcal{I}') \to \mathcal{HC}^2(\mathcal{I})\,.$$

If $\mathcal{HC}^1(\mathcal{I}) = 0$ and $\mathcal{HC}^2(\mathcal{I}) = 0$, we have the exact sequence

$$0 \to \mathcal{HC}^3(\mathcal{I}) \to \mathcal{H}^3(\mathcal{I},\mathcal{I}') \to 0,$$

and thus $\mathcal{HC}^3(\mathcal{I}) = \mathcal{H}^3(\mathcal{I},\mathcal{I}')$, which proves the theorem.

That $\mathcal{HC}^1(\mathcal{I}) = 0$ is straightforward. A derivation $D : \mathcal{I} \to \mathcal{I}'$ satisfies $D(z^k)(z^l) = kD(z)(z^{k+l-1})$, but a cyclic derivation also satisfies $D(z^k)(z^l) = -D(z^l)(z^k)$. Thus $kD(z)(z^{k+l-1}) = -lD(z)(z^{k+l-1})$, which in turn implies that $D(z)(z^{k+l-1}) = 0$. Therefore $D(z^k)(z^l) = 0$ for all $k, l \in \mathbf{N}$: there are no non-trivial cyclic derivations.

The proof that $\mathcal{HC}^2(\mathcal{I}) = 0$ is more involved. We must show that any cyclic 2-cocycle T is the coboundary of a cyclic 1-cochain S. So let T be a cyclic 2-cocycle on $\mathcal{I}$, that is, $T \in \mathcal{ZC}^2(\mathcal{I})$.

We claim that S defined by

$$S(z^k)(z^{N-k}) = \frac{k}{N}\sum_{n=1}^{N-2} T(z^n, z^1)(z^{N-n-1}) - \sum_{n=1}^{k-1} T(z^n, z^1)(z^{N-n-1})\,,$$

where k and N are positive integers such that $k < N$, is in $CC^1(\mathcal{I})$ and that $\delta^1 S = T$. Let us prove our claim.

For i, j, k and N positive integers such that $i + j + k = N$, we consider the following:

$$\begin{aligned}
&S(z^j)(z^{k+i}) - S(z^{i+j})(z^k) + S(z^i)(z^{j+k})\\
&\quad = -\sum_{n=1}^{j-1} T(z^n, z^1)(z^{N-n-1}) + \sum_{n=1}^{i+j-1} T(z^n, z^1)(z^{N-n-1}) - \sum_{n=1}^{i-1} T(z^n, z^1)(z^{N-n-1})\\
&\quad = -\sum_{n=1}^{j-1} T(z^n, z^1)(z^{N-n-1}) + \sum_{n=i}^{i+j-1} T(z^n, z^1)(z^{N-n-1})\\
&\quad = -\sum_{n=1}^{j-1} T(z^n, z^1)(z^{N-n-1}) + \sum_{n=1}^{j-1} T(z^{i+n}, z^1)(z^{N-n-i-1}) + T(z^i, z^1)(z^{N-i-1})\,.
\end{aligned}$$

As T is a 2-cocycle, this is equal to

$$\begin{aligned}
&\sum_{n=1}^{j-1} T(z^i, z^{n+1})(z^{N-n-i-1}) - \sum_{n=1}^{j-1} T(z^i, z^n)(z^{N-n-i}) + T(z^i, z^1)(z^{N-i-1})\\
&\quad = T(z^i, z^j)(z^{N-i-j}) - T(z^i, z^1)(z^{N-i-1}) + T(z^i, z^1)(z^{N-i-1})\\
&\quad = T(z^i, z^j)(z^k)\,.
\end{aligned}$$

Thus we have shown that

$$T(z^i, z^j)(z^k) = S(z^j)(z^{i+k}) - S(z^{i+j})(z^k) + S(z^i)(z^{j+k})\,.$$

If $S \in CC^1(\mathcal{I})$, the last equation implies that $\delta^1 S = T$. Thus we need to show that S is bounded, cyclic and normalized.

The equation above implies $T(z^i, z^j)(z^k) = T(z^j, z^i)(z^k)$. Given that T is cyclic, we have $T(z^i, z^j)(z^k) = T(z^i, z^k)(z^j)$. Let us use this to show that $S(z^k)(z^{N-k}) = -S(z^{N-k})(z^k)$, which implies that S is a cyclic cocycle if S is bounded. We have

$$\sum_{n=k}^{N-2} T(z^n, z^1)(z^{N-n-1}) = \sum_{n=k}^{N-2} T(z^{N-n-1}, z^1)(z^n) = \sum_{n=1}^{N-k-1} T(z^n, z^1)(z^{N-n-1}),$$

and so

$$\begin{aligned} S(z^k)(z^{N-k}) &= \frac{k}{N}\sum_{n=1}^{N-2} T(z^n, z^1)(z^{N-n-1}) - \sum_{n=1}^{k-1} T(z^n, z^1)(z^{N-n-1}) \\ &= \frac{k-N}{N}\sum_{n=1}^{N-2} T(z^n, z^1)(z^{N-n-1}) + \sum_{n=k}^{N-2} T(z^n, z^1)(z^{N-n-1}) \\ &= -\frac{N-k}{N}\sum_{n=1}^{N-2} T(z^n, z^1)(z^{N-n-1}) + \sum_{n=1}^{N-k-1} T(z^n, z^1)(z^{N-n-1}) \\ &= -S(z^{N-k})(z^k). \end{aligned}$$

We can now prove that S is bounded, using a doubling argument. For a given N, let i be such that

$$\left|S(z^i)(z^{N-i})\right| = \max_{k=1,\dots,N-1} \left|S(z^k)(z^{N-k})\right| .$$

As $S(z^k)(z^{N-k}) = -S(z^{N-k})(z^k)$, we can suppose that $i \le N/2$.

If $i = N/2$ then we have $S(z^{N/2})(z^{N/2}) = -S(z^{N/2})(z^{N/2})$, which must then be zero.

If $i < N/2$ then we have

$$T(z^i, z^i)(z^{N-2i}) = S(z^i)(z^{N-i}) - S(z^{2i})(z^{N-2i}) + S(z^i)(z^{N-i}),$$

which implies

$$2\left|S(z^i)(z^{N-i})\right| \le \|T\| + \left|S(z^{2i})(z^{N-2i})\right| .$$

Thus we get that $\max_{k=1,\dots,N-1}\left|S(z^k)(z^{N-k})\right| \le \|T\|$, and so S is bounded.

Given that S is bounded, we can now conclude that $\delta^1 S = T$ and that S is cyclic. It is clear that S is normalized, which completes the proof. □

Remark 2.4. An argument similar to the doubling argument used in the last part of the proof is used to prove Proposition 4.9.

3. Transferring the problem

Our goal is to show that $\mathcal{H}^3(\mathcal{A}, \mathcal{A}') = 0$, and it follows from Theorem 2.3 that this is equivalent to $\mathcal{HC}^3(\mathcal{I}) = 0$. In this section, we show that $\mathcal{HC}^3(\mathcal{I}) = 0$ is equivalent to a problem for some functions defined on points in a simplex which lie on a certain lattice.

To do so, given $T \in C^n(\mathcal{A}, \mathcal{A}')$ and $N \in \mathbf{N}$, we define the function T_N on the integer n-tuples $(i_1, i_2, \dots, i_n)$ such that $i_1 + i_2 + \cdots + i_n \le N$ by

$$(1) \qquad T_N(i_1, i_2, \dots, i_n) = T(z^{i_1}, z^{i_2}, \dots, z^{i_n})(z^{N-(i_1+\cdots+i_n)}).$$

We note that the set of points for which T_N is defined lie on an n-simplex of size N, and we denote the space of such functions by $C^n(\Sigma_N)$.

To obtain a complex, we define the maps $\delta_N^n : C^n(\Sigma_N) \to C^{n+1}(\Sigma_N)$ by

$$\begin{aligned}(\delta_N^n T_N)(i_1, i_2, \ldots, i_n, i_{n+1}) &= T_N(i_2, i_3, \ldots, i_{n+1}) \\ &\quad - T_N(i_1 + i_2, i_3, \ldots, i_{n+1}) + \cdots \\ &\quad + (-1)^n T_N(i_1, i_2, \ldots, i_n + i_{n+1}) \\ &\quad + (-1)^{n+1} T_N(i_1, i_2, \ldots, i_n) .\end{aligned}$$

With this definition, $\left(\delta_N^n, C^n(\Sigma_N)\right)$ forms a complex (that is, $\delta_N^{n+1} \circ \delta_N^n = 0$), and we can define $\mathcal{B}^n(\Sigma_N)$, $\mathcal{Z}^n(\Sigma_N)$ and $\mathcal{H}^n(\Sigma_N)$. It is clear that $\delta_N^n(T_N) = (\delta^n T)_N$, and we can therefore transfer a problem on the complex $\left(\delta^n, C^n(\mathcal{A}, \mathcal{A}')\right)$ to a series of problems on the complexes $\left(\delta_N^n, C^n(\Sigma_N)\right)$.

This process can be reversed, as we now show. Starting with T_N, we define an n-cochain $\tilde{T} \in C^n(\mathcal{A}, \mathcal{A}')$ by

$$\begin{aligned}\tilde{T}(z^{i_1}, z^{i_2}, \ldots, z^{i_n})(z^{i_0}) &= T_N(i_1, i_2, \ldots, i_n) \quad \text{if } \sum_{k=0}^{n} i_k = N; \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise} .\end{aligned}$$

We say that T_N is *cyclic* if $\tilde{T}$ (as defined by the preceding formulae) is cyclic, and we denote the space of all such cyclic T_N by $CC^n(\Sigma_N)$. Similarly, we say that T_N is *normalized* (respectively, *even*, *odd*) if $\tilde{T}$ is normalized (respectively, even, odd).

It follows from the definition that T_N is normalized if it vanishes whenever one of the entries is zero, and T_N is normalized and cyclic if it vanishes on the boundary of the simplex, that is whenever one of the entries is zero or when $i_1 + i_2 + \cdots + i_n = N$.

If we have a uniformly bounded family $\{T_N\}_{N \in I}$ ($I \subseteq \mathbf{N}$), we can define $\tilde{T} \in C^n(\mathcal{A}, \mathcal{A}')$ using essentially the same formulae, namely

$$\begin{aligned}\tilde{T}(z^{i_1}, z^{i_2}, \ldots, z^{i_n})(z^{i_0}) &= T_N(i_1, \ldots, i_n) \quad \text{if } \sum_{k=0}^{n} i_k = N \text{ and } N \in I; \\ &= 0 \qquad\qquad\qquad \text{otherwise} .\end{aligned}$$

The following facts are then easy to prove and are collected in a lemma for ease of reference.

Lemma 3.1. *The following hold.*

1. *T_N is cyclic iff $T_N(a_1, a_2, \ldots, a_n) = (-1)^n T_N(a_0, a_1, \ldots, a_{n-1})$ when $a_0 + a_1 + \cdots + a_n = N$.*
2. *If T_N is cyclic, then $\delta_N^n(T_N)$ is cyclic.*
3. *δ_N^n maps $CC^n(\Sigma_N)$ into $CC^{n+1}(\Sigma_N)$.*
4. *If T_N is normalized, then $\delta_N^n(T_N)$ is normalized.*
5. *If T_N, $N \in \mathbf{N}$, is a uniformly bounded family such that each T_N is cyclic and normalized, then $\tilde{T}$ (as defined above) is cyclic and normalized.*

Proof. By definition, T_N is cyclic if $\tilde{T}$ is cyclic. Hence T_N is cyclic iff for all $a_0 + a_1 + \cdots + a_n = N$ we have

$$\begin{aligned}T_N(a_1, a_2, \ldots, a_n) &= \tilde{T}(z^{a_1}, z^{a_2}, \ldots, z^{a_n})(z^{a_0}) \\ &= (-1)^n \tilde{T}(z^{a_0}, z^{a_1}, \ldots, z^{a_{n-1}})(z^{a_0}) \\ &= (-1)^n T_N(a_0, a_1, \ldots, a_{n-1}),\end{aligned}$$

which proves the first statement.

To prove the second statement, we have: T_N cyclic implies $\tilde{T}$ cyclic by definition; $\tilde{T}$ cyclic implies $\delta^n\tilde{T}$ cyclic, as we mentioned earlier (the cyclic cochains form a subcomplex); $\delta^n\tilde{T}$ cyclic implies $(\delta^n\tilde{T})_N$ cyclic, which easily follows from the definitions; and $\delta^n_N(T_N) = (\delta^n\tilde{T})_N$ implies the result.

The third statement is a simple rewriting of the second.

The fourth statement follows from the definition of $\delta^n T_N$: it is easy to check that if one of the variables i_1, i_2, ... , i_{n+1} is zero, then all of the terms in the definition of $\delta^n T_N(1_1, 1_2, \dots, i_{n+1})$ are zero except for two terms which cancel out.

The fifth statement immediately follows from the definitions of $\tilde{T}$ and T_N. □

From the second part of the lemma, we see that we can define the spaces $\mathcal{BC}^n(\Sigma_N)$, $\mathcal{ZC}^n(\Sigma_N)$ and $\mathcal{HC}^n(\Sigma_N)$ to be the cyclic analogues of $\mathcal{B}^n(\Sigma_N)$, $\mathcal{Z}^n(\Sigma_N)$ and $\mathcal{H}^n(\Sigma_N)$.

Given that we wish to show that $\mathcal{HC}^3(\mathcal{I}) = 0$, let us consider $T \in CC^3(\mathcal{I})$. We define T_N on the integers n-tuples $(i_1, i_2, \dots, i_n)$ such that $i_1 + i_2 + \dots + i_n \le N$ in the following way. We let

$$T_N(i_1, i_2, \dots, i_n) = T(z^{i_1}, z^{i_2}, \dots, z^{i_n})(z^{N-(i_1+\dots+i_n)})$$

if $i_1 + i_2 + \dots + i_n < N$ and $i_k \neq 0$ for $k = 1, \dots, n$, and we let

$$T_N(i_1, i_2, \dots, i_n) = 0$$

otherwise, that is, if one of the variables is zero or if $i_1 + i_2 + \dots + i_n = N$.

If $T \in \mathcal{ZC}^3(\mathcal{I})$, then T_N is a normalized element of $CC^3(\Sigma_N)$. In fact, T_N is a cocycle. To verify this, let a, b, c, d, e be integers such that $a+b+c+d+e = N$. Then

$$\begin{aligned} \delta^3_N T_N(a,b,c,d) &= T_N(b,c,d) - T_N(a+b,c,d) + T_N(a,b+c,d) \\ &\quad - T_N(a,b,c+d) + T_N(a,b,c)\,. \end{aligned}$$

If a, b, c, d and e are all non-zero, then $\delta^3_N T_N(a,b,c,d) = (\delta^3 T)_N(a,b,c,d) = 0$. If a, b, c or d is zero, then it is easy to see that three of the terms on the right-hand side of the equation above are zero and the other two cancel out. If $e = 0$, the middle three terms are zero while the other two cancel out precisely because T_N is cyclic.

Remark 3.2. The argument above works in general to show that if $T \in \mathcal{ZC}^n(\mathcal{I})$, $n \ge 1$, then $T_N \in \mathcal{ZC}^n(\Sigma_N)$.

We can now state the following theorem.

Theorem 3.3. *There exists an absolute constant K such that, given $N \in \mathbf{N}$ and a normalized $\omega \in \mathcal{ZC}^3(\Sigma_N)$, there exists a normalized $\phi \in CC^2(\Sigma_N)$ such that $\delta^2\phi = \omega$ and $\|\phi\| \le K\,\|\omega\|$.*

The proof of this theorem is rather involved and constitutes the whole of Section 4. Before giving this proof, we state the following corollary which, by Theorem 2.3, implies that the third simplicial cohomology group of $l^1(\mathbf{Z}_+)$ vanishes.

Corollary 3.4. $\mathcal{HC}^3(\mathcal{I}) = 0$.

Proof. If $T \in \mathcal{ZC}^3(\mathcal{I}, \mathcal{I}')$, then, for each $N \in \mathbf{N}$, T_N is a normalized element of $\mathcal{ZC}^3(\Sigma_N)$. We can therefore apply Theorem 3.3 to obtain that, for each $N \in \mathbf{N}$, there exists a normalized $\phi_N \in CC^2(\Sigma_N)$ such that $\delta^2_N\phi_N = T_N$ and $\|\phi_N\| \le K\,\|T_N\| \le K\,\|T\|$.

Given that the family ϕ_N is bounded by $K\,\|T\|$ for all $N \in \mathbf{N}$, we can define the corresponding normalized $\tilde{\phi} \in CC^2(\mathcal{I}, \mathcal{I}')$. It is clear that $\delta^2\tilde{\phi} = T$, which proves the result. $\square$

4. Proof of Theorem3.3

The proof of Theorem 3.3 proceeds by the application of several results which form the major part of this section. To simplify the notation, we denote by δ the map δ^2_N.

Proposition 4.1. *Given $N \in \mathbf{N}$ and a normalized $\omega \in \mathcal{ZC}^3(\Sigma_N)$, there exists a normalized $\phi \in CC^2(\Sigma_N)$ such that $\delta\phi = \omega$.*

Proof. The cyclic cocycle $\omega \in \mathcal{ZC}^3(\Sigma_N)$ defines a cyclic 3-cocycle $\tilde{\omega}$ on the polynomial algebra $\mathbf{C}[X]$, with coefficients in $\mathbf{C}[X]'$, by the same formula used to define $\tilde{T}$ from T_N. It is well known that the algebra $\mathbf{C}[X]$, the algebraic dual space, has projective bidimension 1, that is, it has a resolution by biprojective $\mathbf{C}[X]$ bimodules which has length 1. In fact, this resolution is

$$0 \leftarrow \mathbf{C}[X] \leftarrow \mathbf{C}[X] \otimes \mathbf{C}[X] \leftarrow \mathbf{C}[X] \otimes \mathbf{C}[X] \leftarrow 0\,,$$

where the first map is multiplication and the second sends $X^i \otimes X^j$ to $X^{i+1} \otimes X^j - X^i \otimes X^{j+1}$. This shows that the second and higher *algebraic* cohomology groups of $\mathbf{C}[X]$ with coefficients in any bimodules are zero. (These facts can be found in [8].)

It follows from the algebraic Connes-Tzygan exact sequence that $HC^3(\mathbf{C}[X]) \cong HC^1(\mathbf{C}[X])$, but the same argument used in the proof of Theorem 2.3 shows $HC^1(\mathbf{C}[X]) = 0$. Thus $\tilde{\omega}$ is the coboundary of some cyclic ϕ. The function ϕ gives rise to a function $\phi_N \in CC^2(\Sigma_N)$, which has coboundary ω.

The function ϕ_N is cyclic, but it may not be normalized. However, let ϕ'_N be defined by

$$\begin{aligned} \phi'_N(a,b) &= 0 \qquad\qquad \text{if } a = 0, b = 0 \text{ or } \ a + b = N\,; \\ \phi'_N(a,b) &= \phi(a,b) \qquad \text{otherwise,} \end{aligned}$$

where a and b are integers such that $a+b \le N$. Then the following argument shows that ϕ'_N, which is normalized and cyclic, satisfies $\delta\phi'_N = \omega$. Let a, b, c and d be integers such that $a+b+c+d = N$. Then

$$\delta\phi'_N(a,b,c) = \phi'_N(b,c) - \phi'_N(a+b,c) + \phi'_N(a,b+c) - \phi'_N(a,b)\,.$$

If a, b, c and d are all non-zero, then $\delta\phi'_N(a,b,c) = \delta\phi_N(a,b,c) = \omega(a,b,c)$. If a, b or c is zero, then two terms cancel out and the other two are zero. If $d = 0$, then the middle two terms are zero and $\phi'_N(b,c) = \phi'_N(a,b)$ precisely because ϕ'_N is cyclic. Thus, if a, b, c or d is zero, then $\delta\phi'_N(a,b,c) = 0 = \omega(a,b,c)$, as ω is normalized. $\square$

We now prove a technical result which will be important later on. Essentially, this proposition enables us to suppose that N is even.

Proposition 4.2. *Let $\phi \in CC^2(\Sigma_N)$ be such that $\|\delta\phi\| \le M$. Then we can extend ϕ to points (a,b), where a and b are of the form $m/2$ for $m \in \{0,1,2,3,\ldots,2N\}$, in such a way that:*

1. *ϕ is cyclic in the sense that $\phi(a,b) = \phi(b,c)$ if $a+b+c = N$ (where a, b and c are of the form $m/2$ for some integer m);*

2. $\|\delta\phi\| \leq M$, *where*

$$\delta\phi(a,b,c) = \phi(b,c) - \phi(a+b,c) + \phi(a,b+c) - \phi(a,b);$$

3. *the extension of* ϕ *is normalized if the original* ϕ *was.*

Proof. We define $\phi(a,b) = \frac{1}{2}\big(\phi(\lfloor a\rfloor, \lceil b\rceil) + \phi(\lceil a\rceil, \lfloor b\rfloor)\big)$. This is clearly well defined and agrees with the definition of ϕ if a and b are integers.

To prove that ϕ is cyclic we need to show that $\phi(a,b) = \phi(b,c)$ if $a+b+c = N$. We prove this if c is an integer and if a and b are not; the other cases are treated similarly. Then

$$\phi(a,b) = \frac{1}{2}\big(\phi\big(\lfloor a\rfloor, \lceil b\rceil\big) + \phi\big(\lceil a\rceil, \lfloor b\rfloor\big)\big) = \frac{1}{2}\big(\phi(\lceil b\rceil, c) + \phi(\lfloor b\rfloor, c)\big) = \phi(b,c)$$

because ϕ is cyclic for integers and $\lfloor a\rfloor + \lceil b\rceil = \lceil a\rceil + \lfloor b\rfloor = a+b$.

To prove that $\|\delta\phi\| \leq M$, we need to show that

$$|\delta\phi(a,b,c)| = |\phi(b,c) - \phi(a+b,c) + \phi(a,b+c) - \phi(a,b)|$$

is less than or equal to M when a, b and c are such that $a+b+c \leq N$. If a, b and c are integers, we already have this. If not, then we consider the following four equations:

$$\begin{aligned}
\phi(b,c) &= \frac{1}{2}\big(\phi(\lfloor b\rfloor, \lceil c\rceil) + \phi(\lceil b\rceil, \lfloor c\rfloor)\big)\,; \\
-\phi(a+b,c) &= -\frac{1}{2}\big(\phi(\lfloor a+b\rfloor, \lceil c\rceil) + \phi(\lceil a+b\rceil, \lfloor c\rfloor)\big)\,; \\
\phi(a,b+c) &= \frac{1}{2}\big(\phi(\lfloor a\rfloor, \lceil b+c\rceil) + \phi(\lceil a\rceil, \lfloor b+c\rfloor)\big)\,; \\
-\phi(a,b) &= -\frac{1}{2}\big(\phi(\lfloor a\rfloor, \lceil b\rceil) + \phi(\lceil a\rceil, \lfloor b\rfloor)\big)\,.
\end{aligned}$$

If b is an integer and x is either an integer or a half integer, then using $\lfloor b\rfloor = \lceil b\rceil = b$, $\lfloor b+x\rfloor = b + \lfloor x\rfloor$ and $\lceil b+x\rceil = b + \lceil x\rceil$ we can regroup the first, third, fifth and seventh terms on the right-hand side of the equations to obtain

$$\begin{aligned}
|\delta\phi(a,b,c)| &= \frac{1}{2}\Big|\phi(b,\lceil c\rceil) - \phi(\lfloor a\rfloor + b, \lceil c\rceil) + \phi(\lfloor a\rfloor, b + \lceil c\rceil) - \phi(\lfloor a\rfloor, b) \\
&\quad + \phi(b,\lfloor c\rfloor) - \phi(\lceil a\rceil + b, \lfloor c\rfloor) + \phi(\lceil a\rceil, b + \lfloor c\rfloor) - \phi(\lceil a\rceil, b)\Big| \\
&\leq \frac{1}{2}(M+M) = M\,.
\end{aligned}$$

If b is not an integer, we have $\lceil b+x\rceil = \lceil b\rceil + \lfloor x\rfloor$ and $\lfloor b+x\rfloor = \lfloor b\rfloor + \lceil x\rceil$. Regrouping the first, third, sixth and eighth term yields

$$\begin{aligned}
|\delta\phi(a,b,c)| = \frac{1}{2}\Big|&\phi(\lfloor b\rfloor, \lceil c\rceil) - \phi(\lceil a\rceil + \lfloor b\rfloor, \lceil c\rceil) + \phi(\lceil a\rceil, \lfloor b\rfloor + \lceil c\rceil) - \phi(\lceil a\rceil, \lfloor b\rfloor) \\
&+ \phi(\lceil b\rceil, \lfloor c\rfloor) - \phi(\lfloor a\rfloor + \lceil b\rceil, \lfloor c\rfloor) + \phi(\lfloor a\rfloor, \lceil b\rceil + \lfloor c\rfloor) - \phi(\lfloor a\rfloor, \lceil b\rceil)\Big| \\
\leq \frac{1}{2}&(M+M) = M\,.
\end{aligned}$$

Finally, to check that the extension is normalized, we need $\phi(a,b) = 0$ when $a+b = N$, even if both are non-integer. However, this value is just the average of the values at $(\lceil a\rceil, \lfloor b\rfloor)$ and $(\lfloor a\rfloor, \lceil b\rceil)$: as each of these pairs is a pair of integers adding up to N, the value of ϕ at these points is zero. □

Even though this will not be used, we remark that the above process can be repeated and ϕ extended to all dyadic rationals in the simplex. Then, as each extension is piecewise linear and so continuous, the extension proceeds to points with real coordinates in the simplex, maintaining the cyclic, normalization, cocycle and coboundary properties (with a suitably extended definition of ω).

The next theorem is key: it shows that we can modify $\phi \in CC^2(\Sigma_N)$ in such a way that it vanishes on the diagonal, that is, $\phi(a,a) = 0$ for all a. This will be used to show that the norm of such a normalized ϕ is bounded by a constant which does not depend on N.

Theorem 4.3. *Given a normalized $\phi \in CC^2(\Sigma_N)$, there exists a normalized $\phi' \in CC^2(\Sigma_N)$ such that $\delta\phi' = \delta\phi$ and $\phi'(a,a) = 0$ for all a.*

Proof. We add a coboundary $\delta_N^1\psi$ to ϕ to obtain the desired condition. We ensure that the function ψ is normalized and cyclic, so that $\delta_N^1\psi$ is also normalized and cyclic (by Lemma 3.1). The function ψ is defined in terms of the diagonal values of ϕ extended to $\mathbf{Z}$ by periodicity and in such a way that it is an odd function. We define

$$\tilde{\phi}(x) = \begin{cases} +\phi(+x \bmod N, +x \bmod N) & \text{if } x \bmod N \le N/2; \\ -\phi(-x \bmod N, -x \bmod N) & \text{if } x \bmod N > N/2. \end{cases}$$

This function is defined for all $x \in \mathbf{N}$, and it has the following useful properties:

$$\begin{aligned} \tilde{\phi}(x+kN) &= \tilde{\phi}(x)\,; \\ \tilde{\phi}(-x) &= -\tilde{\phi}(x)\,; \\ \tilde{\phi}(x) &= \phi(x,x) \quad \text{for } 0 \le x \le N/2\,. \end{aligned}$$

We now define

$$\psi(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \tilde{\phi}(2^n x)\,,$$

where the infinite sum converges because ϕ is bounded. We define $\phi' = \phi - \delta_N^1\psi$. For $x \le N/2$, we have

$$\begin{aligned} \psi(2x) &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \tilde{\phi}(2^{n+1}x) = 2\sum_{n=0}^{\infty} \frac{1}{2^{n+1+1}} \tilde{\phi}(2^{n+1}x) \\ &= 2\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \tilde{\phi}(2^n x) = 2\psi(x) - \tilde{\phi}(x)\,. \end{aligned}$$

Thus, for $x \le N/2$, we have

$$\phi'(x,x) = \phi(x,x) - \psi(x) + \psi(x+x) - \psi(x) = \phi(x,x) - \tilde{\phi}(x) = 0.$$

This shows ϕ' vanishes on the diagonal. We now need to show that ϕ' is still cyclic and normalized, which will follow (by Lemma 3.1) from the fact that ψ is. We have

$$\begin{aligned} \psi(N-x) &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \tilde{\phi}(2^n(N-x)) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \tilde{\phi}(2^n(-x)) \\ &= \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} \tilde{\phi}(2^n x) = -\psi(x)\,, \end{aligned}$$

and thus ψ, hence ϕ, is cyclic. Clearly $\psi(0) = \phi(0,0) = 0$ and $\psi(N) = \psi(N-N) = 0$, and so ψ, and hence $\delta\phi$, is normalized. The proof is complete. ∎

The next proposition shows that ϕ can be expressed as the sum of an even and an odd function in a convenient way.

Proposition 4.4. *Let $\phi \in CC^2(\Sigma_N)$ be normalized and such that $\phi(a,a) = 0$ for all a. Then we can write $\phi = \phi_+ + \phi_-$, where ϕ_+ and ϕ_- are normalized elements of $CC^2(\Sigma_N)$ which vanish on the diagonal and are respectively even and odd. Furthermore, if $\|\delta\phi\| \le M$, then $\|\delta\phi_+\| \le M$ and $\|\delta\phi_-\| \le M$.*

Proof. We can write any ϕ as a sum $\phi = \phi_+ + \phi_-$, where

$$\begin{aligned} \phi_+(x,y) &= \frac{1}{2}(\phi(x,y) + \phi(y,x)), \\ \phi_-(x,y) &= \frac{1}{2}(\phi(x,y) - \phi(y,x)). \end{aligned}$$

This decomposition is simply the rewriting for $C^2(\Sigma_N)$ of the general decomposition for $C^n(\mathcal{A},\mathcal{A}')$ presented in Section 1. As $k_n = 4$, we have the usual definition for odd and even, that is, $\phi_+(a,b) = \phi_+(b,a)$ and $\phi_-(a,b) = -\phi_-(b,a)$.

It is clear that ϕ_+ and ϕ_- are normalized and vanish on the diagonal. Lemma 1.1 implies that if ϕ is cyclic, then so are ϕ_+ and ϕ_-. It also shows that if $\|\delta\phi\| \le M$, then $\|\delta\phi_+\| \le M$ and $\|\delta\phi_-\| \le M$. □

The next lemma shows that the odd function ϕ_- is close to linear in the second variable.

Lemma 4.5. *Let $\phi_- \in CC^2(\Sigma_N)$ be odd and such that $\|\delta\phi_-\| \le M$. Then*

$$|\phi_-(a,b+c) - \phi_-(a,b) - \phi_-(a,c)| \le \frac{3M}{2}.$$

Proof. We have

$$\begin{aligned} 3M &\ge |\delta\phi_-(a,b,c) - \delta\phi(c,a,b) + \delta\phi_-(b,c,a)| \\ &= |\phi_-(b,c) - \phi_-(a+b,c) + \phi_-(a,b+c) - \phi_-(a,b) \\ &\quad -\phi_-(a,b) + \phi_-(c+a,b) - \phi_-(c,a+b) + \phi_-(c,a) \\ &\quad +\phi_-(c,a) - \phi_-(b+c,a) + \phi_-(b,c+a) - \phi_-(b,c)| \\ &= 2\,|\phi_-(a,b+c) - \phi_-(a,b) - \phi_-(a,c)|, \end{aligned}$$

which immediately yields the result. □

To get a bound on $\|\phi_-\|$, we use the previous lemma with the following.

Lemma 4.6. *Let f be a real valued function defined on $\{0,1,\dots,L\}$ such that*

$$|f(x+y) - f(x) - f(y)| \le K.$$

Then f is within $3K$ of the linear function $g(x) = \frac{f(L)x}{L}$; that is, $|f(x) - g(x)| \le 3K$.

Proof. Without loss of generality, we can suppose $f(L) = 0$ (by replacing f by $f-g$) and $f(x_0) = \|f\|_\infty = M \ge 0$ for some x_0. Thus we only need to show that $M \le 3K$.

If $x_0 \le L/2$, then $|f(2x_0) - 2f(x_0)| \le K$. Given that $|f(2x_0)| \le M$, we get

$$|f(2x_0) - 2f(x_0)| = |f(2x_0) - 2M| \ge M,$$

and so $M \le K$.

If $x_0 \geq L/2$, let $x_1 = L - x_0$. We then have

$$|f(L) - f(x_0) - f(x_1)| \leq K,$$

and so $|f(x_1) + M| \leq K$. Thus

$$-M - K \leq f(x_1) \leq -M + K\,.$$

We also have $|f(2x_1) - 2f(x_1)| \leq K$, and so

$$2f(x_1) - K \leq f(2x_1) \leq 2f(x_1) + K\,.$$

In particular, we get that $f(2x_1) \leq -2M + 3K$. If $M \geq \frac{3K}{2}$ then $-2M + 3K \leq 0$, and so

$$|f(2x_1)| \geq |-2M + 3K| = 2M - 3K\,.$$

However we also must have $M \geq |f(2x_1)|$, which gives $M \geq 2M - 3K$. Hence $M \leq 3K$. □

Note: The previous technical lemma obviously holds in other settings, for instance if f is defined on the interval $[0, 1]$.

Proposition 4.7. *Let $\phi_- \in CC^2(\Sigma_N)$ be odd, normalized and such that $\|\delta\phi_-\| \leq M$. Then $\|\phi_-\| \leq \frac{9\sqrt{2}}{2}M$.*

Proof. Let $f_a(x)$ be the real part of $\phi_-(a, x)$. Then f_a is a real-valued function defined on $\{0, 1, \ldots, N - a\}$ and, from Lemma 4.5, is such that

$$|f_a(x + y) - f_a(x) - f_a(y)| \leq \frac{3M}{2}\,.$$

Hence, by Lemma 4.6 and the fact that $f_a(N - a) = 0$, we have

$$|f_a(x)| \leq 3 \cdot \frac{3M}{2} = \frac{9M}{2}\,.$$

The absolute value of the imaginary part of $\phi_-(a, x)$ is also bounded by $\frac{9M}{2}$, and thus $|\phi_-(a, x)| \leq \frac{9\sqrt{2}}{2}M$ for $x \in \{0, 1, \ldots, N - a\}$. Letting a vary, we get the result. □

We now proceed to show that $\|\phi_+\|$ is also bounded by some constant which does not depend on N.

Lemma 4.8. *Let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \leq M$. Let a, b be such that $2a + 2b \leq N$. Then*

$$|\phi_+(2a, 2b) - 2\phi_+(a, b)| \leq 3M\,.$$

Proof. Applying $\|\delta\phi\| \leq M$ to points a, b, b and a, b, $a+b$ together with $\phi_+(b, b) = \phi_+(a + b, a + b) = 0$ yields

$$\begin{aligned} |-\phi_+(a + b, b) + \phi_+(a, 2b) - \phi_+(a, b)| &\leq M\,; \\ |\phi_+(b, a + b) + \phi_+(a, 2b + a) - \phi_+(a, b)| &\leq M\,. \end{aligned}$$

Adding those two equations and using $\phi_+(a + b, b) = \phi_+(b, a + b)$ gives

$$|\phi_+(a, 2b) + \phi_+(a, 2b + a) - 2\phi_+(a, b)| \leq 2M.$$

Applying $\|\delta\phi\| \leq M$ to points a, a, $2b$ together with $\phi_+(a, a) = 0$, we get

$$|\phi_+(a, 2b) - \phi_+(2a, 2b) + \phi_+(a, 2b + a)| \leq M\,.$$

The last two inequalities combine to give the desired result; that is,

$$|\phi_+(2a, 2b) - 2\phi_+(a, b)| \le 3M .$$

□

Proposition 4.9. *Let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \le M$. Let (a, b) be such that $|\phi_+(a, b)| = \|\phi_+\|$. If either $a + b \le N/2$, $a \ge N/2$ or $b \ge N/2$, then $\|\phi_+\| \le 3M$.*

Proof. If $a + b \le N/2$ then Lemma 4.8 implies

$$\|\phi_+\| = |\phi_+(a, b)| \le |\phi_+(2a, 2b) - 2\phi_+(a, b)| \le 3M .$$

If $a \ge N/2$ then for $c = N - (a + b)$ we have, using ϕ_+ cyclic,

$$\phi_+(a, b) = \phi_+(b, c) .$$

Given that $b + c = N - a \le N/2$, we have from the first part that $|\phi_+(b, c)| \le 3M$. The proof for $b \ge N/2$ is identical. □

We now need to treat the case where $|\phi_+(a, b)| = \|\phi_+\|$ for a and b such that $a + b \ge N/2$, $a \le N/2$ and $b \le N/2$.

Lemma 4.10. *Let N be even, and let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \le M$. If $a + b = \frac{N}{2}$, then*

$$|\phi_+(a, b)| \le \frac{M}{2} .$$

Proof. Applying $\|\delta\phi\| \le M$ to points a, a, b with $\phi_+(a, a) = 0$ gives

$$|\phi_+(a, b) - \phi_+(2a, b) + \phi_+(a, b + a)| \le M .$$

Using the fact that ϕ_+ is even and cyclic together with $2a + 2b = N$, we have

$$\begin{aligned} \phi_+(2a, b) &= \phi_+(b, b) = 0 , \\ \phi_+(a, b + a) &= \phi_+(b, a) = \phi_+(a, b) \end{aligned}$$

and so $|\phi_+(a, b)| \le \frac{M}{2}$. □

Using Lemma 4.10 requires N even, which explains the need for Proposition 4.2. We also note that it is possible to prove the previous lemma (with a slightly different bound) without using the cyclic property.

Proposition 4.11. *Let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \le M$. Let (b, c) be such that $b \le \frac{N}{2}$ and $c \le \frac{N}{2}$. Then*

$$\left|\phi_+(b, c) - \phi_+(\frac{N}{2} - b, \frac{N}{2} - c)\right| \le 2M .$$

Proof. Let $a = \frac{N}{2} - b$ and $d = \frac{N}{2} - c$. Applying $\|\delta\phi\| \le M$ to points a, b, c gives

$$\left|\phi_+(b, c) - \phi_+(\frac{N}{2}, c) + \phi_+(a, b + c) - \phi_+(a, b)\right| \le M .$$

Using ϕ_+ cyclic and Lemma 4.10 gives the two inequalities

$$\begin{aligned} \left|\phi_+(\frac{N}{2}, c)\right| &= \left|\phi_+(c, \frac{N}{2} - c)\right| \le \frac{M}{2} ; \\ |\phi_+(a, b)| &\le \frac{M}{2} . \end{aligned}$$

We also have (again from cyclic) $\phi_+(a, b+c) = \phi_+(d, a)$. Combining these results with the first inequality (and using ϕ_+ even) gives $|\phi_+(b,c) + \phi_+(a,d)| \leq 2M$. □

Corollary 4.12. *Let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \leq M$. Let (a,b) be such that $a \leq \frac{N}{2}$, $b \leq \frac{N}{2}$ and $a+b \geq \frac{N}{2}$, and let $c = \frac{N}{2} - a$ and $d = \frac{N}{2} - b$. Then*

$$|\phi_+(c,d) - 2\phi_+(a,b)| \leq 7M\,.$$

Proof. As $2c + 2d \leq N$, we have from Lemma 4.8

$$|\phi_+(2c, 2d) - 2\phi_+(c,d)| \leq 3M\,.$$

From Proposition 4.11 and using the fact that ϕ_+ is even, we also have

$$|2\phi_+(a,b) - 2\phi_+(c,d)| \leq 4M\,.$$

These two inequalities immediately give the desired result. □

Corollary 4.13. *Let $\phi_+ \in CC^2(\Sigma_N)$ be even, normalized, vanishing on the diagonal and such that $\|\delta\phi_+\| \leq M$. Suppose that $|\phi_+(a,b)| = \|\phi_+\|$. If $a \leq \frac{N}{2}$, $b \leq \frac{N}{2}$ and $a + b \geq \frac{N}{2}$, then $\|\phi_+\| \leq 7M$.*

Proof. We have the conditions of Corollary 4.12, and so

$$|\phi_+(a,b)| \leq |\phi_+(c,d) - 2\phi_+(a,b)| \leq 7M\,.$$

□

Proof of Theorem 3.3. Using the results of this section, we can now give a proof of Theorem 3.3.

Proof. Let ω be a normalized element of $\mathcal{ZC}^3(\Sigma_N)$. By Proposition 4.1 there exists a normalized $\phi \in CC^2(\Sigma_N)$ such that $\delta\phi = \omega$.

If N is odd, we extend the definition of ϕ to points with half-integer coordinates in the manner described in Proposition 4.2, which ensures that ϕ is still cyclic and normalized. We then extend the definition of ω to points with half-integer coordinates so that $\delta\phi = \omega$ on those points as well; that is, we let

$$\omega(x,y,z) = \phi(y,z) - \phi(x+y,z) + \phi(x,y+z) - \phi(x,y).$$

It follows from the second part of Proposition 4.2 that this extension does not increase the norm of ω, and it is clear that the extension of ω is still normalized.

Thus ϕ and ω are defined on half integers, and we can reinterpret them as normalized functions in $CC^2(\Sigma_{2N})$ and $\mathcal{ZC}^3(\Sigma_{2N})$ such that $\delta\phi = \omega$. This shows that we can, without loss of generality, suppose that N is even.

By Theorem 4.3, we can further suppose that we have a normalized $\phi \in CC^2(\Sigma_N)$ such that $\phi(a,a) = 0$ and $\delta\phi = \omega$. Proposition 4.4 implies that ϕ can be expressed as the sum of an odd and an even function which, by Propositions 4.7, 4.9 and Corollary 4.13, have norm bounded by $\frac{9\sqrt{2}}{2}\,\|\omega\|$ and $7\,\|\omega\|$ respectively.

Thus we have obtained the result with $K = 7 + \frac{9\sqrt{2}}{2}$. □

5. Conclusion

It is our conjecture that all higher order simplicial cohomology group of $l^1(\mathbf{Z}_+)$ vanish, and it would be interesting to find a general argument which would yield this result. However, our initial attempts at doing this have not succeeded.

Another interesting question is to determine the simplicial cohomology groups of $l^1(\mathbf{Z}_+^k)$. We believe that it is possible to obtain these, using a Künneth type argument, once we know all simplicial cohomology groups of $l^1(\mathbf{Z}_+)$. In particular, we conjecture that $\mathcal{H}^n(l^1(\mathbf{Z}_+^k), l^\infty(\mathbf{Z}_+^k)) = 0$ for $n > k$.

Let us note here that, contrary to what is stated in [3, p. 114], it is not true that the second cohomology group of $l^1(\mathbf{Z}_+^k)$ vanishes for each $k \in \mathbf{N}$. For $n > 1$, we consider the elements of $l^1(Z_+^k)$ as functions on a polydisc. The first 2 variables are denoted z and w. We denote partial differentiation with respect to z and w by f_z, f_w respectively. Consider the 2-cochain defined by

$$\phi(f,g)(h) = (f_z(0)g_w(0) - f_w(0)g_z(0))h(0).$$

It is a simple computation to see this is a 2-cocycle. Note that it is odd and non-zero. Any coboundary $\delta\psi(f,g)(h) = \psi(g)(hf) - \psi(fg)(h) + \psi(f)(gh)$ must be even, so ϕ cannot be a coboundary and so the second cohomology group cannot vanish if $k > 1$.

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References

[1] W.G. Bade, P.C. Curtis Jr., and H.G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, *Proc. London Math. Soc.* (3), 55 (1987), 359–377. MR **88f:**46098

[2] F.F. Bonsall and J. Duncan, *Complete normed algebras,* Springer-Verlag, New York, 1973. MR **54:**11013

[3] H. G. Dales and J. Duncan, Second order cohomology in groups of some semigroup algebras, Banach Algebras '97, Proc. 13th. Internat. Confer. on Banach Algebras, 101–117, Walter de Gruyter, Berlin, 1998. MR **99m:**46168

[4] A. Ya. Helemskii, The homology of Banach and topological algebras. Kluwer Academic Publishers, Dordrecht, 1989. MR **92d:**46178

[5] A. Ya. Helemskii, Banach cyclic (co)homology and the Connes-Tzygan exact sequence, *J. London Math. Soc.*, (2) 46, (1992), 449-462. MR **94a:**46105

[6] B.E. Johnson, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.,* **127** (1972). MR **51:**11130

[7] B.E. Johnson, Higher-dimensional weak amenability, *Studia Math.*, 123 (1997), 2, 117–134. MR **98f:**46046

[8] D.G. Northcott, Introduction to homological algebra, Cambridge University Press, 1962. MR **22:**9523

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